The General Relativistic Precession of Mercury's Perihelion and the Corrections to Newton's Gravitational Potential and Force

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Abstract:

A calculation of the general relativistic apsidal or perihelion precession of planet Mercury is presented, based on the Schwarzschild gravitational metric field tensor and the mathematical perturbation theory and yielding the same result as in Albert Einstein's 1916 paper. Both the classical and the general relativistic contributions to Mercury's precession result in breaking Noether's symmetry of Kepler's first law by which a planetary closed orbit exactly repeats itself periodically, thus the Laplace-Runge-Lenz vector is not anymore conserved. A clear calculation of the general-relativity corrections to both Newton's gravitational potential and force is also shown.

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I THE SCHWARZSCHILD METRIC AND THE GEODESIC EQUATIONS

An exact solution to the 6 independent Einstein field equations describing an elliptical orbital motion is the Schwarzschild metric field tensor, whose squared line element is

$$ds^2 = c^2\left(1-\frac{2\mu}{r}\right)dt^2 - \left(1-\frac{2\mu}{r}\right)^{-1}dr^2 - r^2\,d\theta^2 - r^2\sin^2\theta\,d\phi^2, \tag{1}$$

where

$$\mu = GM/c^2. \tag{2}$$

Defining the squared Lagrangian for massive bodies as

$$\begin{aligned} L &= g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu \\ &= c^2. \end{aligned} \tag{3}$$

where

$$\dot{x}^\mu = dx^\mu/d\tau, \tag{4}$$

and τ is the affine time parameter, we get, using (1),

$$L = c^2\left(1-\frac{2\mu}{r}\right)\dot{t}^2 - \left(1-\frac{2\mu}{r}\right)^{-1}\dot{r}^2 - r^2\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2\right). \tag{5}$$

The Euler-Lagrange equations are

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right) - \frac{\partial L}{\partial x^\mu} = 0. \tag{6}$$

where

$$\mu = 0, 1, 2, 3.$$

By means of (5) and (6),we get the 4 spacetime geodesic equations [6]

$$\left(1-\frac{2\mu}{r}\right)\dot{t} = k, \tag{7}$$

$$\left(1-\frac{2\mu}{r}\right)^{-1}\ddot{r} + \frac{\mu c^2}{r^2}\dot{t}^2 - \left(1-\frac{2\mu}{r}\right)^{-2}\frac{\mu}{r^2}\dot{r}^2 - r\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2\right) = 0, \tag{8}$$

$$\ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \sin\theta\cos\theta\,\dot{\phi}^2 = 0, \tag{9}$$

$$r^2\sin^2\theta\,\dot{\phi} = h. \tag{10}$$

where h, the angular momentum per unit rest mass, and k, the total energy per unit rest energy, are constants.

For a celestial massive body moving in the plane

$$\theta = \pi/2,$$

the set of the 4 spacetime geodesic equations reduces to

$$\left(1-\frac{2\mu}{r}\right)\dot{t} = k, \tag{11}$$

$$\left(1-\frac{2\mu}{r}\right)^{-1}\ddot{r} + \frac{\mu c^2}{r^2}\dot{t}^2 - \left(1-\frac{2\mu}{r}\right)^{-2}\frac{\mu}{r^2}\dot{r}^2 - r\dot{\phi}^2 = 0, \tag{12}$$

$$r^2\dot{\phi} = h. \tag{13}$$

(3) yields

$$c^2\left(1-\frac{2\mu}{r}\right)\dot{t}^2 - \left(1-\frac{2\mu}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\phi}^2 = c^2. \tag{14}$$

Substituting (11) and (13) into (14), we obtain

$$\dot{r}^2+\frac{h^2}{r^2}\left(1-\frac{2GM}{c^2 r}\right)-\frac{2GM}{r}=c^2(k^2-1). \qquad (15)$$

In (15), letting

$$r\to\infty \text{ and } \dot{r}=0,$$

one gets

$$k^2=1.$$

By substituting

$$\frac{dr}{d\tau}=\frac{dr}{d\phi}\frac{d\phi}{d\tau}=\frac{h}{r^2}\frac{dr}{d\phi} \qquad (17)$$

into (15), we get

$$\left(\frac{h}{r^2}\frac{dr}{d\phi}\right)^2+\frac{h^2}{r^2}=c^2(k^2-1)+\frac{2GM}{r}+\frac{2GMh^2}{c^2r^3}. \qquad (18)$$

With the substitution

$$u\equiv 1/r, \qquad (19)$$

(18) becomes

$$\left(\frac{du}{d\phi}\right)^2+u^2=\frac{c^2}{h^2}(k^2-1)+\frac{2GMu}{h^2}+\frac{2GMu^3}{c^2}. \qquad (20)$$

By differentiating (20) with respect to ϕ , we obtain a differential equation describing the elliptical motion of a massive body in Schwarzschild geometry and given by

$$\frac{d^2u}{d\phi^2}+u=\frac{GM}{h^2}+\frac{3GM}{c^2}u^2. \qquad (21)$$

For a circular motion,

$$r=\text{constant},$$

$$\dot{r}=\ddot{r}=0,$$

thus,

$$u=1/r=\text{constant},$$

and (21) becomes

$$u=\frac{GM}{h^2}+\frac{3GM}{c^2}u^2. \qquad (22)$$

II EQUATION OF MOTION FOR PLANETARY ORBITS IN NEWTONIAN GRAVITY

The classical-mechanics Lagrangian of a celestial object of mass m influenced by the central gravitational field force of a massive body M in the equatorial plane

$$\theta=\pi/2$$

is given by

$$L=\tfrac{1}{2}m(\dot{r}^2+r^2\dot{\phi}^2)+\frac{GMm}{r}. \qquad (23)$$

From the Euler-Lagrange equations, we get

$$r^2\dot{\phi}=h, \qquad (24)$$

$$\ddot{r}=\frac{h^2}{r^3}-\frac{GM}{r^2}. \qquad (25)$$

Additionally, the motionally constant total energy is given by

$$E=\frac{1}{2}m(\dot{r}^2+r^2\dot{\phi}^2)-\frac{GMm}{r}. \qquad (26)$$

By plugging (19) and (24) into (26) and using

$$\frac{du}{dt}=\frac{du}{d\phi}\frac{d\phi}{dt}=\frac{h}{r^2}\frac{du}{d\phi}, \qquad (27)$$

we obtain the Newtonian equation of motion for general elliptical planetary orbits

$$\frac{d^2u}{d\phi^2}+u=\frac{GM}{h^2}. \qquad (28)$$

III SOLUTION TO THE NEWTONIAN ORBITAL EQUATION

The function

$$u_A=\frac{GM}{h^2} \qquad (29)$$

is a particular solution of the nonhomogeneous linear second-order differential equation (28), identified as the reciprocal of the semi-latus rectum , and represents a circular orbit.

The general solution of the homogeneous equation of (28) is

$$u_B=C\cos(\phi-\phi_0), \qquad (30)$$

where C is a constant of motion.

The general solution of (28) is

$$\begin{aligned} u &= u_A+u_B \\ &= \frac{GM}{h^2}+C\cos(\phi-\phi_0). \end{aligned} \qquad (31)$$

Defining the eccentricity of the orbital trajectory as

$$e=\frac{Ch^2}{GM}, \qquad (32)$$

and knowing that the eccentricity is defined in function of the magnitude of Laplace-Runge-Lenz vector $\vec{A}$ as

$$e=\frac{A}{mk}, \qquad (33)$$

where

$$k=GMm, \qquad (34)$$

we obtain

$$C=\frac{A}{h^2m^2}. \qquad (35)$$

Setting

$$\phi_0 = 0,$$

one gets

$$u = \frac{GM}{h^2}(1 + e\cos\phi). \quad (36)$$

IV SOLUTION TO THE EINSTEINIAN ORBITAL EQUATION

In (21),

$$\frac{3GMu^2}{c^2} \ll \frac{GM}{h^2}. \quad (37)$$

In fact

$$\frac{3GMu^2}{c^2} \div \frac{GM}{h^2} = \frac{3GM}{c^2}\left(\frac{1}{r}\right)^2 \div \frac{GM}{(rv)^2} = 3v^2/c^2, \quad (38)$$

which is in the case of Mercury, the fastest solar planet,

$$\approx 10^{-7}.$$

Let

$$\epsilon = \frac{3(GM)^2}{h^2c^2} \ll 1. \quad (39)$$

Thus, (21) can be written as

$$u'' + u = \frac{GM}{h^2} + \epsilon\left(\frac{h^2u^2}{GM}\right). \quad (40)$$

(38) is a non-linear and nonhomogeneous second-order ordinary differential equation where

$$\epsilon = \ll 1.$$

Therefore, (40) can be solved by means of the well-established perturbative method, and the general solution of (38) has the form

$$u = \epsilon^0 u_0 + \epsilon u_1 + O(\epsilon^2). \\ = u_0 + \epsilon u_1 + O(\epsilon^2). \quad (41)$$

By setting ϵ equal to zero in both (40) and (41), we get the Newtonian classical orbital equation of motion whose general solution is (36),i.e.,

$$u = u_0 = \frac{GM}{h^2}(1 + e\cos\phi). \quad (42)$$

Consequently, as a first-order approximation, the Newtonian solution u_0 is the zeroth-order solution of the general-relativistic orbital equation of motion (21) .

By differentiating (41) twice and substituting the result into (40), we get (43)

$$u_\circ'' + u_\circ - \frac{GM}{h^2} + \epsilon(u_1'' + u_1 - \frac{h^2u_\circ^2}{GM}) + O(\epsilon^2) = 0$$

or

$$\epsilon^0(u_\circ'' + u_\circ - \frac{GM}{h^2}) + \epsilon(u_1'' + u_1 - \frac{h^2u_\circ^2}{GM}) + O(\epsilon^2) = 0. \quad (44)$$

Equating both sides of (44), we get

$$u_\circ'' + u_\circ - \frac{GM}{h^2} = 0, \quad (45)$$

$$u_1'' + u_1 - \frac{h^2u_\circ^2}{GM} = 0. \quad (46)$$

By substituting (42) into (46), one gets

$$u_1'' + u_1 = \frac{GM}{h^2}(1 + 2ecos\phi + e^2cos^2\phi) \quad (47)$$

or

$$u_1'' + u_1 = \frac{m}{h^2}(1 + \frac{1}{2}e^2) + \frac{2me}{h^2}cos\phi + \frac{me^2}{2h^2}cos2\phi, \quad (48)$$

which can be written as

$$u_1'' + u_1 = \mathrm{D} + \mathrm{E}\,cos\phi + \mathrm{F}\,cos2\phi. \quad (49)$$

By choosing the general solution of (49) to be of the form

$$u_1 = A + B\phi sin\phi + Ccos2\phi \quad (50)$$

and differentiating u_1 twice, we obtain

$$u_1'' + u_1 = (A) + (2B)cos\phi + (-3C)cos2\phi. \quad (51)$$

by simple comparison between (48) and (51), one gets

$$A = \frac{GM}{h^2}(1 + \frac{1}{2}e^2), \quad (52)$$

$$B = \frac{GMe}{h^2}, \quad (53)$$

$$C = -\frac{GMe^2}{6h^2}. \quad (54)$$

Therefore,

$$u_1 = \frac{GM}{h^2}(1 + \frac{1}{2}e^2) + \frac{GMe}{h^2}\phi sin\phi - \frac{GMe^2}{6h^2}cos2\phi, \quad (55)$$

and the general solution up to the first order in ϵ of (40) is

$$u = u_\circ + \frac{GM\epsilon}{h^2}[1 + e\phi sin\phi + e^2(\frac{1}{2} - \frac{1}{6}cos2\phi)]. \quad (56)$$

By substituting (42) into (56), we get

$$u = \frac{GM}{h^2} + \frac{GM}{h^2}e\cos\phi + \frac{GM\epsilon}{h^2} + \frac{GM\epsilon}{h^2}e\phi sin\phi + \frac{GM\epsilon}{h^2}\frac{e^2}{2} - \frac{GMe^2}{6h^2}\epsilon\, cos2\phi. \quad (57)$$

All terms with ϵ can be neglected except

$$\frac{GM\epsilon}{h^2}e\phi sin\phi, \quad (58)$$

since the value of ϕ continuously increases with time as the planet moves on its elliptical path. Hence, (57) becomes

$$u = \frac{GM}{h^2}(1 + ecos\phi + \epsilon e\phi sin\phi). \quad (59)$$

Using

$$\cos[\phi(1-\epsilon)] = \cos\phi\cos\epsilon\phi + \sin\phi\sin\epsilon\phi \approx \cos\phi + \epsilon\phi\sin\phi \quad \text{for} \quad \epsilon \ll 1, \tag{60}$$

(56) becomes

$$u = \frac{GM}{h^2}[1 + e\cos[\phi(1-\epsilon)]]. \tag{61}$$

V APSIDAL ANGLE AND CLOSED ORBITS

In (25),

$$-\frac{GM}{r^2},$$

represents an attractive gravitational central force per unit mass. For any per-unit-mass conservative central force, (25) is generalized to

$$\ddot{r} - \frac{h^2}{r^3} = F_r(r). \tag{62}$$

Let

$$r(t) = a + \rho(t), \tag{63}$$

where a is the radius of a circular orbit and ρ is a small perturbation such as

$$\rho(t) \ll a. \tag{64}$$

Substituting (63) into (62), we get, up to the first-order perturbation ,

$$\ddot{\rho} - \frac{h^2}{a^3}\left(1 - 3\frac{\rho}{a}\right) = F_r(a) + \frac{dF_r(a)}{dr}\rho. \tag{65}$$

For a circular orbit, (62) becomes

$$-\frac{h^2}{r^3} = F_r(r). \tag{66}$$

Plugging (66) into (65), we obtain

$$\ddot{\rho} + \left[\frac{-3F_r(a)}{a} - F_r'(a)\right]\rho = 0,$$

where

$$F_r'(a) = \frac{dF_r(a)}{dr}. \tag{67}$$

For a simple harmonic motion, the term in square brackets is positive , i.e.,

$$F_r(a) + \frac{a}{3}F_r'(a) < 0, \tag{68}$$

and stands for the squared orbital or angular frequency. Thus, the circular orbit of radius a is stable to small perturbations. Additionally, the nearly circular orbit of the radial distance $r(t)$ is elliptically bounded since $r(t)$ oscillates between a fixed maximum value and a fixed minimum value called apsides, where, in the case of the Sun, the minimum apsis is the perihelion and the maximum apsis is the aphelion.

The orbital time period of oscillation is given by

$$T = 2\pi\left[\frac{-3F_r(a)}{a} - F_r'(a)\right]^{-\frac{1}{2}}. \tag{69}$$

An attractive central power-law force has the form

$$F_r(r) = -\kappa r^n, \tag{70}$$

where n is an integer and

$$\kappa > 0.$$

Substituting (70) into (68), we obtain

$$-\kappa a^n - \frac{\kappa n a^n}{3} < 0. \tag{71}$$

(71) yields

$$n > -3. \tag{72}$$

The apsidal angle,φ, is the angle between two consecutive apsides, a periastron and an apastron, thus the half-cycle angular half-period. Additionally, the time taken by the orbit to sweep one apsidal angle is clearly half of the orbital time period.

Since the angular velocity ω or $\dot{\phi}$ is the angle through which the massive object turns in one second, the apsidal angle is given by

$$\varphi = \frac{T}{2}\omega. \tag{73}$$

But the angular momentum per unit rest mass, h, is a constant of motion , and the radial distance r is nearly constant for nearly circular orbits; hence we can write, using (66) and (24),

$$\omega \simeq \frac{h}{a^2} = \left[-\frac{F_r(a)}{a}\right]^{\frac{1}{2}}. \tag{74}$$

By means of (69) and (74), the apsidal angle per half-revolution becomes

$$\varphi = \pi\left[3 + \frac{aF_r'(a)}{F_r(a)}\right]^{-\frac{1}{2}}, \tag{75}$$

or,

$$\varphi = \frac{\pi}{(3+n)^{1/2}}. \tag{76}$$

A closed orbit is an elliptical orbit that exactly repeats itself within every orbital time period, and whose apsidal angle is a rational fraction of 2π radians, which is the case of an inverse-square force law , where $n = -2$, such as Newton's gravitational law and the electrostatic law, and a linear force

law, where $n = 1$, such as Hook's law. Hence, (72) and (76) are in total agreement with Bertrand's theorem where only elliptically bounded orbits influenced by the inverse-square law and the radial-harmonic-oscillator law are stably closed. The converse is also true since all closed orbits are elliptical. Therefore, the apsidal angle for an elliptical orbit obeying an inverse-square central-force law is π radians.

VI GENERAL-RELATIVISTIC PRECESSION OF MERCURY

Apsidal or perihelion precession of a planetary orbit is the rotation of the direction joining the apsides, which is the elliptical major axis, around an elliptical focus, which is the Sun in the case of the solar system, in the plane of the orbit.

Since Kepler's first inverse-squared force law yields closed orbits where the perihelion or aphelion remains constant in time, it fails to indicate any planetary precession, which is certainly in contradiction with the observational fact that the perihelion, the aphelion, and the whole elliptical orbit rotationally precess in the direction of the planet's rotation round the focal Sun.

For Mercury, the observed dynamical precession is found to be $575''$per century, where $532''$per century are nonrelativistic and due to the gravitational disturbances of the Sun as well as 6 outer planets—Venus, Earth, Mars, Jupiter, and Saturn—and can be estimated in classical mechanics. But the remaining $43''$per century cannot be account for in Newton's theory of gravity [28] [4].

In 1915 and 1916, Albert Einstein, by means of his theory of general relativity, could accurately explain and solve the centurial $43''$discrepancy, which is found to be purely general relativistic and due the elliptical motion of Mercury in the gravitationally curved spacetime of the Sun.

In fact, the general-relativistic orbital solution (61) yields an angular radian period

$$\frac{2\pi}{1-\epsilon}, \tag{77}$$

which is 2 times the apsidal angle with

$$\frac{2\pi}{1-\epsilon} \approx 2\pi(1+\epsilon), \tag{78}$$

thus an irrational fraction of 2π, yielding an open elliptical orbit.

Since 2π is the value of the apsidal angle for a complete orbital cycle of a Keplerian closed orbit, the additional term $2\pi\epsilon$ represents the general relativistic correction to the Keplerian apsidal angle for a full revolution. Hence, by using (39), the general relativistic contribution to the planetary precession per revolution is

$$\Delta\varphi = 2\pi\epsilon = \frac{6\pi(GM)^2}{h^2c^2}. \tag{79}$$

For nearly circular orbits, such as Mercury's orbit, the radial distance r is almost constant and equal to the semi-latus rectum, thus (25) becomes

$$\begin{aligned} h^2 &= GMr \\ &= GM(1-e^2)\,a. \end{aligned} \tag{80}$$

From Kepler's third law,

$$T^2 = \frac{4\pi^2 a^3}{GM}. \tag{81}$$

Plugging both (80) and (81) into (79), one gets [1]

$$\Delta\varphi = 24\pi^3 \frac{a^2}{T^2c^2\,(1-e^2)}, \tag{82}$$

which is exactly the same as in Albert Einstein's 1916 paper titled "The Foundation of the Generalised Theory of Relativity."

For Mercury, the orbital period in Earth days is

$$T = 88 \text{ days}, \tag{83}$$

the semi-major axis is

$$a = 5.8 \times 10^{10}\,\text{m}, \tag{84}$$

and the eccentricity is

$$e = 0.2, \tag{85}$$

and using

$$M_\odot = 2 \times 10^{30}\,\text{kg} \tag{86}$$

as the solar mass, the general relativistic precessional contribution per revolution amounts to $0.104''$.

In an Earth century, Mercury makes 415 revolutions, yielding a centurial general relativistic precession equal to $43''$.

In celestial mechanics, every Keplerian orbit is associated with two vectors that are constant throughout the planetary motion, thus conserved: the orbital- angular-momentum and the Laplace-Runge-Lenz vector, which is defined by

$$\begin{aligned} \vec{A} &= \vec{p} \times \vec{L} - mk\frac{\vec{r}}{r} \\ &= mk\vec{e}, \end{aligned} \tag{87}$$

where $\vec{e}$ is the eccentricity vector.

The LRL vector always points towards the perihelion along the direction of the symmetry axis of the orbit, lies in the plane of orbital motion, and, in consequence, fixes geometrically the direction of the major axis—which passes through both the perihelion and the aphelion—and yields a Noetherian symmetry by which a Keplerian orbit repeats itself after every orbital period of time.

Any departure from the inverse-squared force law breaks up the symmetry and the conservation of the Laplace-Runge-Lenz vector; as a result, the LRL vector starts to rotate slowly round the Sun, causing the elliptical orbit of planet Mercury to precess.

VII CORRECTIONS TO NEWTON'S GRAVITATIONAL FORCE AND POTENTIAL

The nonrelativistic or classical contribution to the precession of a planetary orbit follows Newton's inverse-square force law, thus does not result in any correction to the gravitational force, unlike the general relativistic precessional contribution, which is given by (79) or (82) and, when purely considered, leads ,by means of (78),(79),and (80), to a purely relativistic precessional apsidal angle per cycle written as

$$2\varphi = 2\pi(1+\epsilon) = 2\pi\left(1+\frac{3GM}{rc^2}\right), \tag{88}$$

and to a correction to Newton's gravitational force per unit mass given by

$$GM\frac{\epsilon}{r^2}, \tag{89}$$

such that the classical gravitational force per unit mass in the case of the Sun-Mercury system becomes [12]

$$\begin{aligned} F_r &= -\frac{GM_\odot}{r^2} - GM_\odot\frac{\epsilon}{r^2} \\ &= -\frac{GM_\odot}{r^2} - \frac{3(GM_\odot)^2}{r^3c^2}, \\ &= -\frac{GM_\odot}{r^2} - 3GM_\odot\frac{h^2}{r^4c^2}, \end{aligned} \tag{90}$$

which can be demonstrated using (75).

In fact, the derivation of (90) with respect to r yields

$$F_r' = 2\frac{GM_\odot}{r^3} + 12\,GM_\odot\frac{h^2}{r^5c^2}. \tag{91}$$

Also, with a simple calculation, we get

$$\frac{rF_r'}{F_r} = -2\left(1+\frac{6h^2}{r^2c^2}\right)\left(1+\frac{3h^2}{r^2c^2}\right)^{-1}. \tag{92}$$

Since

$$\frac{3h^2}{r^2c^2} \ll 1, \tag{93}$$

(92) becomes

$$\frac{rF_r'}{F_r} = -2\left(1+\frac{6h^2}{r^2c^2}\right)\left(1-\frac{3h^2}{r^2c^2}\right). \tag{94}$$

Taking into account that

$$\frac{18h^4}{r^4c^4} \ll 1, \tag{95}$$

(94) can be written as

$$\frac{rF_r'}{F_r} = -2\left(1+\frac{3h^2}{r^2c^2}\right). \tag{96}$$

Plugging (96) into (75) for nearly circular planetary orbits, we get the general relativistic apsidal angle for a half-revolution

$$\varphi = \pi\left(1+\frac{3h^2}{r^2c^2}\right). \tag{97}$$

multiplying (97) by 2, we obtain the general relativistic apsidal angle for a complete revolution

$$2\varphi = 2\pi\left(1+\frac{3h^2}{r^2c^2}\right) = 2\pi\left(1+\frac{3GM}{rc^2}\right), \tag{98}$$

which is identical to (88) with a full-revolution general relativistic precession equal to

$$\frac{6\pi GM}{rc^2}, \tag{99}$$

which is certainly also identical to (76).

Defining the classical gravitational potential to be the work done by a conservative gravitational-field force to bring a unit mass from infinity, where the gravitational potential approaches zero, to a point at a radial distance r, we can write

$$V = \int_\infty^r \vec{F}\,d\vec{r}, \tag{100}$$

and substituting (80) and (90) into (100), we obtain

$$V = -\frac{GM_\odot}{r} - \frac{3}{2}\frac{(GM_\odot)^2}{r^2c^2}, \tag{101}$$

where

$$-\frac{3}{2}\frac{(GM_{\odot})^2}{r^2c^2}, \tag{102}$$

is the general-relativity correction to Newton's gravitational potential.

Unlike Newton' theory of gravity, Einstein's general relativity accurately estimates not only the additional precession due to curved spacetime but also the classical precessional contribution caused by other planets, which is calculated within the frame of general relativity in the case of weak gravitational fields where the curvature of spacetime is of very small value, yielding Newtonian gravity.

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